

Nonleptonic kaon decays: theory vs. experiment

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Abstract

A review of the present experimental status of the rates and the Dalitz-plot slope parameters in CP conserving $K \rightarrow 2\pi$ and $K \rightarrow 3\pi$ decays is given. The corresponding isospin amplitudes have been determined by a common fit to the recent experimental data and have been used as an input to a consequent fit based on the constraints from $O(p^4)$ chiral lagrangian. It has been found that the constraints of the chiral fit are well satisfied by the experimental data, allowing to estimate the weak coupling constants and to predict the quadratic amplitudes in $K \rightarrow 3\pi$ decays and the one-loop strong rescattering phases. The consistency of the obtained weak coupling constants with weak resonance models has been also considered.

1 Introduction

Quantum Chromodynamics (QCD) is the established theory for describing strong quark-quark interactions. However, the theoretical study of kaon decays is hindered by the nonperturbative behavior of QCD in the low-energy domain. One of the approaches successfully applied to the study of low-energy processes is chiral perturbation theory (ChPT) [1, 2]. It is based on the chiral symmetry properties of QCD and the concept of the effective field theory. An important point in the application of ChPT is the existence of coupling constants which can not be calculated from first principles and therefore must be extracted phenomenologically from the experimental data.

In the present paper we report an update of the analysis of CP conserving $K \rightarrow 2\pi$ and $K \rightarrow 3\pi$ decays, performed in [3]. The update is caused by necessity to take into account the recent measurements of $\Gamma(K_S \rightarrow \pi^+\pi^-\pi^0)$ and the Dalitz plot parameters in $K \rightarrow 3\pi$ decays, carried out by the CPLEAR [4], the HYPERON [5] and the NA48 [6] collaborations.

In section 2 we present the results from a phenomenological fit of $K \rightarrow 2\pi$ and $K \rightarrow 3\pi$ isospin amplitudes to the recent experimental data. In section 3 we consider the estimation of the isospin amplitudes in terms of $O(p^4)$ chiral lagrangian and summarize the results from a fit of the weak coupling constants. In section 4 we briefly discuss the consistency of the obtained results with weak resonance models. Section 5 contains some concluding remarks.

2 Phenomenological fit

Using the usual isospin decomposition CP conserving $K \rightarrow 2\pi$ amplitudes can be expressed in the following way¹:

$$\begin{aligned} A(K^0 \rightarrow \pi^+\pi^-) &= -\frac{1}{\sqrt{3}}A_0 + \frac{1}{\sqrt{6}}A_2 \\ A(K^0 \rightarrow \pi^0\pi^0) &= \frac{1}{\sqrt{3}}A_0 + \sqrt{\frac{2}{3}}A_2 \\ A(K^+ \rightarrow \pi^+\pi^0) &= \frac{\sqrt{3}}{2}A_2 \end{aligned} \tag{1}$$

where $A_0 = ia_{1/2}e^{i\delta_0}$ and $A_2 = -ia_{3/2}e^{i\delta_2}$ represent transitions with $\Delta I = 1/2, 3/2$, respectively.

Due to the small available phase space in $K \rightarrow 3\pi$ decays their amplitudes can be expanded in powers of Dalitz plot variables u and v :

¹assuming $\Delta I \leq 3/2$

$$\begin{aligned}
A(K_L \rightarrow \pi^+\pi^-\pi^0) &= (\alpha_1 + \alpha_3) - (\beta_1 + \beta_3)u \\
&+ (\zeta_1 - 2\zeta_3)(u^2 + v^2/3) + (\xi_1 - 2\xi_3)(u^2 - v^2/3) \\
A(K_L \rightarrow \pi^0\pi^0\pi^0) &= -3(\alpha_1 + \alpha_3) - 3(\zeta_1 - 2\zeta_3)(u^2 + v^2/3) \\
A(K^+ \rightarrow \pi^+\pi^+\pi^-) &= (2\alpha_1 - \alpha_3) + (\beta_1 - \beta_3/2 + \sqrt{3}\gamma_3)u \\
&+ 2(\zeta_1 + \zeta_3)(u^2 + v^2/3) - (\xi_1 + \xi_3 - \xi'_3)(u^2 - v^2/3) \\
A(K^+ \rightarrow \pi^0\pi^0\pi^+) &= -(\alpha_1 - \alpha_3/2) + (\beta_1 - \beta_3/2 - \sqrt{3}\gamma_3)u \\
&- (\zeta_1 + \zeta_3)(u^2 + v^2/3) - (\xi_1 + \xi_3 + \xi'_3)(u^2 - v^2/3) \\
A(K_S \rightarrow \pi^+\pi^-\pi^0) &= (2/\sqrt{3})\gamma_3v - (4/3)\xi'_3uv
\end{aligned} \tag{2}$$

where

$$u = \frac{s_3 - s_0}{M_{\pi^\pm}^2}, \quad v = \frac{s_2 - s_1}{M_{\pi^\pm}^2}, \quad s_i = (P_K - P_{\pi_i})^2, \quad s_0 = \frac{(s_1 + s_2 + s_3)}{3} \tag{3}$$

and P_K, P_{π_i} are the four-momenta of the kaon and i -th² pion. The subscripts 1 and 3 in (2) correspond to $\Delta I = 1/2, 3/2$ transitions, respectively. Neglecting the strong rescattering final state phases and assuming CP invariance all the isospin amplitudes $\alpha_1, \alpha_3, \beta_1, \beta_3, \gamma_3, \zeta_1, \zeta_3, \xi_1, \xi_3$ and ξ'_3 are real.

Until now the experimentally measured event distributions for CP conserving $K \rightarrow 3\pi$ decays have been analyzed in terms of the Dalitz plot slopes g, h, k :

$$|A(K \rightarrow 3\pi)|^2 \sim 1 + gu + hu^2 + kv^2. \tag{4}$$

The recent experimental data [7] on the decay widths and the Dalitz plot slopes are reported in Table 1³.

The phenomenological fit of $K \rightarrow 3\pi$ isospin amplitudes to the measured decay widths and Dalitz plot slopes was introduced in [8] and was significantly improved in [3] by including the $\Delta I = 3/2$ quadratic amplitudes ζ_3, ξ_3 and ξ'_3 . We have redone the fit by adding the measurements of $\Gamma(K_S \rightarrow \pi^+\pi^-\pi^0)$ and the quadratic Dalitz plot slope k in $K^+ \rightarrow \pi^0\pi^0\pi^+$. The amplitudes $a_{1/2}, a_{3/2}$ and the relative phase $(\delta_2 - \delta_0)$ between them have been evaluated using only the measured $K \rightarrow 2\pi$ decay widths. The results for the isospin amplitudes are shown in the first column of Table 2. Due to small isospin breaking effects the obtained value of -64.6° for the relative phase between $a_{3/2}$ and $a_{1/2}$ are not reliable and has not been used in the further analysis. The relatively high value of χ^2 of the fit is due entirely to an inconsistency of

²index 3 refers to the “odd” pion

³We have included the value of $(-6.1 \pm 0.9_{stat} \pm 0.5_{syst}) \times 10^{-3}$ for the quadratic Dalitz plot slope parameter h in the $K_L \rightarrow 3\pi^0$ decay, measured by the NA48 collaboration [6]

the precisely measured $K_L \rightarrow 3\pi$ and $K^\pm \rightarrow 3\pi$ decay widths. Contrariwise, all the measured Dalitz plot slopes and $\Gamma(K_S \rightarrow \pi^+\pi^-\pi^0)$ seems to be quite consistent.

We have also tried to extract the strong rescattering phase of the amplitudes β from the available data. In order to do this one have to replace the real amplitudes β in (2) with

$$\beta \rightarrow \beta e^{i\delta_\beta} \approx \beta(1 + i\delta_\beta), \quad (5)$$

where δ_β is real [9]. Since the phases of the constant and quadratic amplitudes are expected to be considerably smaller than δ_β [10], they have been neglected. The phase δ_β has been introduced into the phenomenological fitting procedure as a freely varying parameter. As a result we have obtained $|\delta_\beta| < 0.3$ (68% CL) while the isospin amplitudes have remained almost unchanged.

3 Theoretical estimations within $O(p^4)$ ChPT

To the lowest order $O(p^2)$, CP conserving weak chiral Lagrangian is a sum of $(8_L, 1_R)$ and $(27_L, 1_R)$ operators [11]:

$$\mathcal{L}_W^{(2)} = c_2 \text{Tr} \lambda_6 L_\mu L^\mu + c_3 t_{ik}^{jl} \left(\text{Tr} Q_j^i L_\mu \right) \left(\text{Tr} Q_l^k L^\mu \right), \quad (6)$$

where $(Q_j^i)_{ab} = \delta_{ib}\delta_{ja}$ are the projectors in flavor space, $\lambda_6 = Q_3^2 + Q_2^3$, t_{ik}^{jl} are defined to select the 27-plet part of the interaction, $L_\mu = iU^\dagger \partial_\mu U$ is the left-handed weak current, $U = e^{(i\Phi/F_\pi)}$, $F_\pi = 92.4 \text{ MeV}$ is the pion decay constant and Φ is the pseudoscalar meson $SU(3)$ matrix:

$$\Phi = \sqrt{2} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & -\pi^+ & -K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & -K^0 \\ K^- & -\bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}. \quad (7)$$

The lowest order chiral Lagrangian corresponds to “tree” diagrams which contain one weak vertex and gives the following contributions to the $\Delta I = 1/2$ isospin amplitudes [3]:

$$\begin{aligned} \Re e A_0 &= \frac{\sqrt{6}}{F_\pi^2 F_K} (M_K^2 - M_\pi^2) \left(c_2 - \frac{2}{3} c_3 \right) \\ \Re e \alpha_1 &= \frac{1}{3 F_\pi^3 F_K} M_K^2 \left(c_2 - \frac{2}{3} c_3 \right) \\ \Re e \beta_1 &= \frac{-1}{F_\pi^3 F_K} M_\pi^2 \left(c_2 - \frac{2}{3} c_3 \right) \end{aligned} \quad (8)$$

and to the $\Delta I = 3/2$ ones:

$$\begin{aligned}
\Re A_2 &= \frac{-20}{\sqrt{3}F_\pi^2 F_K} (M_K^2 - M_\pi^2) c_3 \\
\Re \alpha_3 &= \frac{20}{9F_\pi^3 F_K} M_K^2 c_3 \\
\Re \beta_3 &= \frac{5}{3F_\pi^3 F_K} \frac{M_\pi^2 (5M_K^2 - 14M_\pi^2)}{(M_K^2 - M_\pi^2)} c_3 \\
\Re \gamma_3 &= \frac{-15}{2\sqrt{3}F_\pi^3 F_K} \frac{M_\pi^2 (3M_K^2 - 2M_\pi^2)}{(M_K^2 - M_\pi^2)} c_3
\end{aligned} \tag{9}$$

Thus the coupling constants c_2 and c_3 of the $\Delta I = 1/2, 3/2$ operators are determined by the two well-measured $K \rightarrow 2\pi$ isospin amplitudes $a_{1/2}$ and $a_{3/2}$. $SU(3)$ symmetry breaking is taken into account by fixing the kaon decay constant F_K to its physical value of 113 MeV .

Now let's consider the corrections to the isospin amplitudes due to the one-loop contributions from $\mathcal{L}_S^{(2)} \times \mathcal{L}_W^{(2)}$ and the tree-level contributions from both $O(p^4)$ weak and strong chiral lagrangians. Following the analysis in [3, 12], we have factorized $K \rightarrow 2\pi$ and $K \rightarrow 3\pi$ isospin amplitudes in the following way:

$$A_i = A_i^{(2)} + A_i^{(4)}, \tag{10}$$

where $A_i^{(2)}$ are the lowest order values defined by equations (8), (9) and $A_i^{(4)}$ are the corresponding next-to-lowest corrections. Furthermore, the latter can be decomposed as:

$$A_i^{(4)} = A_i^{loop}(\mu) + A_i^{wk.ct.}(\mu) + A_i^{st.ct.}(\mu), \tag{11}$$

where $A_i^{loop}(\mu)$, $A_i^{wk.ct.}(\mu)$, $A_i^{st.ct.}(\mu)$ are the one-loop contributions and the tree-level contributions from the counterterms of $O(p^4)$ weak and strong lagrangians, respectively. The parameter μ is the renormalization scale on which, in principle, $A_i^{(4)}$ must be independent.

Since the one-loop contributions $A_i^{loop}(\mu)$ arise from the lowest order weak lagrangian, they are proportional to the coupling constants c_2 and c_3 and are conveniently expressed in terms of the so-called reduced amplitudes $a_i^{(8)}$ and $a_i^{(27)}$:

$$A_i^{loop} = \left(c_2/F_\pi^2\right) a_i^{(8)} + \left(c_3/F_\pi^2\right) a_i^{(27)}. \tag{12}$$

For the calculation of the one-loop contributions we have used the values of the reduced amplitudes for $\mu = M_\eta$ obtained in [3] and presented in Table 3.

As it was shown in [3, 12], without external fields and neglecting corrections of order M_π^2/M_K^2 the contributions from weak counterterms can be expressed in terms of only 7 linear combinations⁴ K_i , $i = (1 \div 7)$, of weak cou-

⁴the explicit form of K_i can be found in [3]

pling constants. On the other side, the contributions from the counterterms of the strong lagrangian are described by 5 out of 10 well-known coupling constants L_i , $i = (1 \div 5)$ (Table 4) [2, 13]. The counterterm contributions to the isospin amplitudes are:

$$\begin{aligned}
\Re e A_0 &= \frac{-2\sqrt{2}}{3\sqrt{3}F_\pi^2 F_K} M_K^4 (K_1 + K_4) \\
\Re e A_2 &= \frac{-20}{3\sqrt{3}F_\pi^2 F_K} M_K^4 K_4 \\
\Re e \alpha_1 &= \frac{-2}{27F_\pi^3 F_K} M_K^4 [(K_1 + K_4 - K_2 + 2K_5) \\
&\quad + \frac{4(3c_2 - 2c_3)}{F_\pi^2} (4L_1 + 4L_2 + 2L_3)] \\
\Re e \alpha_3 &= \frac{20}{27F_\pi^3 F_K} M_K^4 [(K_4 + 2K_5) \\
&\quad - \frac{8c_3}{F_\pi^2} (4L_1 + 4L_2 + 2L_3)] \\
\Re e \beta_1 &= \frac{1}{9F_\pi^3 F_K} M_K^2 M_\pi^2 [(-2K_1 - 2K_4 + K_3 + \frac{16}{27}K_5 + \frac{35}{27}K_6 - \frac{13}{9}K_7) \\
&\quad + \frac{8(3c_2 - 2c_3)}{F_\pi^2} (-2L_1 + L_2 - L_3 + 12L_4)] \\
\Re e \beta_3 &= \frac{5}{18F_\pi^3 F_K} \frac{M_K^4 M_\pi^2}{(M_K^2 - M_\pi^2)} [(10K_4 + 4K_6 + 3K_7) \\
&\quad + \frac{64c_3}{F_\pi^2} (2L_1 - L_2 + L_3 - 12L_4)] \\
\Re e \gamma_3 &= \frac{-5}{4\sqrt{3}F_\pi^3 F_K} \frac{M_K^4 M_\pi^2}{(M_K^2 - M_\pi^2)} [(6K_4 + K_7) \\
&\quad + \frac{8c_3}{F_K^2} \frac{M_K^2}{M_\pi^2} (-L_3 + 12L_4 + 6L_5)] \\
\Re e \zeta_1 &= \frac{-1}{6F_\pi^3 F_K} M_\pi^4 [(K_2 - 2K_5) \\
&\quad - \frac{8(3c_2 - 2c_3)}{F_\pi^2} (2L_1 + 2L_2 + L_3)] \\
\Re e \zeta_3 &= \frac{5}{3F_\pi^3 F_K} M_\pi^4 [K_5 \\
&\quad - \frac{8c_3}{F_\pi^2} (2L_1 + 2L_2 + L_3)] \\
\Re e \xi_1 &= \frac{-1}{6F_\pi^3 F_K} M_\pi^4 [(K_3 + \frac{16}{27}K_5 + \frac{35}{27}K_6 - \frac{13}{9}K_7) \\
&\quad - \frac{8(3c_2 - 2c_3)}{F_\pi^2} (2L_1 - L_2 + L_3)] \\
\Re e \xi_3 &= \frac{-5}{24F_\pi^3 F_K} M_\pi^4 [(4K_6 + 3K_7) \\
&\quad + \frac{64c_3}{F_\pi^2} (2L_1 - L_2 + L_3)] \\
\Re e \xi_3' &= \frac{15}{8F_\pi^3 F_K} M_\pi^4 [K_7 \\
&\quad - \frac{8c_3}{F_K^2} \frac{M_K^2}{M_\pi^2} L_3] .
\end{aligned} \tag{13}$$

Using the twelve isospin amplitudes in the first column of Table 2 and requiring that all the nine relative phases of $K \rightarrow 3\pi$ amplitudes be zero within errors of 15° (compatible with the upper bound of $|\delta_\beta|$), we have fitted the coupling constants c_2 , c_3 and the counterterms K_i . The fitting procedure have shown that $K_1(K_4)$ is strongly correlated(anticorrelated) with $c_2(c_3)$. Thus, neglecting small terms of order M_π^2/M_K^2 in (8) and (9), the coupling constants K_1 and K_4 can be safely absorbed by a proper redefinition of the coupling constants c_2 and c_3 :

$$c_2 \rightarrow c_2' = c_2 - \frac{2}{9}M_K^2 K_1, \quad c_3 \rightarrow c_3' = c_3 + \frac{1}{3}M_K^2 K_4. \quad (14)$$

In this way, the contributions $A_i^{(2)}$, A_i^{loop} and $A_i^{st.ct.}$ are expressed in terms of c_2' and c_3' while the weak counterterm contributions $A_i^{wk.ct.}$ are determined by K_2 , K_3 , K_5 , K_6 and K_7 . The results of the fit are summarized in Table 5 and in the second column of Table 2. Obviously, the experimentally measured isospin amplitudes are in a excellent agreement within the applied chiral scheme. Moreover, the weak contributions of the quadratic amplitudes ζ_1 , ζ_3 , ξ_1 , ξ_3 , ξ_3' are proportional to the weak contributions of the constant and linear amplitudes and therefore are independent of the weak couplings K_i . Thus, the good consistency between the measured and predicted values of the quadratic amplitudes justifies the applicability of the chiral approach at $O(p^4)$.

Based on the fitted values⁵ of c_2' , c_3' and K_i and taking into account the imaginary parts of the one-loop contributions from Table 3, one can predict the phases of $K \rightarrow 3\pi$ amplitudes. The obtained phases of the constant and linear amplitudes are given in Table 6. The imaginary parts of the transition amplitudes were expanded in Dalitz plot variables and calculated numerically. Therefore, considering the values in the table one have to be aware that the phases of the amplitudes β include the kinematically dependent phases of the constant amplitudes α estimated at the lowest order in momentum.

4 Comparison with weak resonance models

The results from the ChPT fit allow to make some remarks of the validity of weak resonance models. In what follows we will consider weak deformation and factorization models [14]. Both models rely on the hypothesis that the strong chiral Lagrangian dominates the features of the $\Delta S = 1$ effective lagrangian. Due to the saturation of the strong couplings of $O(p^4)$ by resonance exchange, the considered models can be also treated as models for the resonance contributions to the weak couplings. As it was shown in [14], the resonance model leads to a scale independent prediction $K_2 - K_3 \simeq 2.6 \times 10^{-9}$, which seems to be compatible with the values in Table 5. A further assumption that only vector resonance contribute implies $K_1 = K_2 = 0$ and $K_3 \neq 0$. Even taking into account the scale dependence⁶ of K_2 , the vector resonance model is far from describing the data. Now lets consider the prediction

⁵the fit has been repeated by excluding the requirement for the smallness of the relative isospin phases

⁶the scale dependencies of the weak coupling can be found in [14]

$$-K_1 \simeq K_2 = K_3 \simeq k_f (1.7 \times 10^{-9}) , \quad (15)$$

which comes from the weak deformation and the factorization models. The scale factor k_f in (15) is equal to 1/2 for the weak deformation model and is around unity for the factorization model. Contrariwise, the obtained values for the couplings K_2 and K_3 require $k_f \approx 4$. The relatively large uncertainties of the weak coupling constants and their sizable scale dependence can reduce this estimation by a factor of 2, which unfortunately remains sufficiently far from the preferred by the factorization model value of ≈ 1 .

5 Conclusions

The applied ChPT approach establish certain relations between $K \rightarrow 2\pi$ amplitudes and the constant and the linear amplitudes in $K \rightarrow 3\pi$ decays. Moreover, the quadratic $K \rightarrow 3\pi$ amplitudes are strongly correlated to the precisely measured constant and linear ones and therefore are independent of the weak coupling constants. The fit to the experimentally measured isospin amplitudes have shown that the chiral constraints are well satisfied not only for the $\Delta I = 1/2$ amplitudes, but also for the $\Delta I = 3/2$ ones. This fact leads to the conclusion of the validity of the chiral scheme in the description of the nonleptonic kaon decays.

In addition to these achievements, the presented analysis has two main disadvantages. The first one is that we have omitted the radiative corrections to $K \rightarrow 3\pi$ amplitudes. As it was shown in [9], these corrections may introduce significant effects especially in the case of $K^+ \rightarrow \pi^+\pi^+\pi^-$ decay. The second disadvantage is neglecting the strong rescattering phases in the estimation of $K \rightarrow 3\pi$ amplitudes. The latter is due to the fact that the experimental information in CP conserving $K \rightarrow 3\pi$ decays is insufficient to provide these phases. From the other side, the real counterterms have been fitted to the assumed real isospin amplitudes and as a consequence the obtained values of the strong rescattering phases are suppressed. Therefore, it would be more desirable to fix the phases, for example, in precise time-interferometry experiments [10] and to use the measurements as an input to the fitting procedure.

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Table 1: Experimental data for the widths and the Dalitz plot slopes in CP conserving $K \rightarrow 3\pi$ decays.

Decay	$\Gamma (s^{-1}) \times 10^{-6}$	$g \times 10^1$	$h \times 10^2$	$k \times 10^2$
$K_L \rightarrow \pi^+ \pi^- \pi^0$	2.43 ± 0.04	6.78 ± 0.08	7.6 ± 0.6	0.99 ± 0.15
$K_L \rightarrow \pi^0 \pi^0 \pi^0$	4.08 ± 0.06		-0.50 ± 0.23	
$K^+ \rightarrow \pi^+ \pi^+ \pi^-$	4.52 ± 0.04	-2.154 ± 0.035	1.2 ± 0.8	-1.01 ± 0.34
$K^+ \rightarrow \pi^0 \pi^0 \pi^+$	1.40 ± 0.03	6.52 ± 0.31	5.7 ± 1.8	1.97 ± 0.54
$K_S \rightarrow \pi^+ \pi^- \pi^0$	0.0036 ± 0.0013			

Table 2: Values of the isospin amplitudes in units of 10^{-8} , obtained from the phenomenological fit to the experimental data (first column) and from the fit within $O(p^4)$ ChPT (second column).

	Experiment	$O(p^4)$
$a_{1/2}, keV$	0.4665 ± 0.0010	0.4665
$a_{3/2}, keV$	0.02116 ± 0.00007	0.02115
α_1	92.12 ± 0.34	92.11
α_3	-6.41 ± 0.44	-6.97
β_1	-26.73 ± 0.39	-26.76
β_3	-2.26 ± 0.44	-2.17
γ_3	2.89 ± 0.28	2.98
ζ_1	-0.38 ± 0.19	-0.51
ζ_3	-0.09 ± 0.10	-0.008
ξ_1	-1.80 ± 0.29	-1.66
ξ_3	0.17 ± 0.15	0.07
ξ'_3	-0.57 ± 0.41	-0.15
$\chi^2/n.d.f$	14.2/5	13.9/14

Table 3: Values of the reduced one-loop amplitudes $a_i^{(8)}$ and $a_i^{(27)}$ for the renormalization scale $\mu = M_\eta$.

	$\Re a_i^{(8)}$	$\Im a_i^{(8)}$	$\Re a_i^{(27)}$	$\Im a_i^{(27)}$
A_0	1.79	2.22	-1.26	-1.48
A_2	-	-	-1.94	4.63
α_1	2.31	1.00	-1.21	-0.67
α_3	-	-	22.5	6.70
β_1	-1.07	0.50	0.70	0.33
β_3	-	-	-0.63	-2.19
γ_3	-	-	-3.58	1.12
ζ_1	-0.027	0.019	0.013	-0.012
$\hat{\zeta}_3$	-	-	0.035	0.066
ξ_1	-0.115	0	0.088	0
ξ_3	-	-	-0.126	0
ξ'_3	-	-	0.294	0.430

Table 4: Values of the strong counterterm coupling constants L_i for the normalization scale $\mu = M_\eta$ in units of 10^{-3} .

L_1	L_2	L_3	L_4	L_5
0.6 ± 0.3	1.75 ± 0.3	-3.5 ± 1.1	0.0 ± 0.5	2.2 ± 0.5

Table 5: Values of the redefined coupling constants c'_2 , c'_3 and the counterterms K_i in units of 10^{-9} obtained from full fit to the experimentally measured isospin amplitudes. The errors are from the experimentally measured amplitudes (first column of Table 2) and from the uncertainties in the strong couplings L_i (Table 4), respectively.

c'_2/F_π^2	65.47 ± 0.14
c'_3/F_π^2	-0.828 ± 0.003
K_2	$6.9 \pm 0.1 \pm 2.2$
K_3	$7.0 \pm 0.7 \pm 2.0$
K_5	$-0.016 \pm 0.003 \pm 0.010$
K_6	$-0.12 \pm 0.08 \pm 0.02$
K_7	$-0.17 \pm 0.06 \pm 0.001$

Table 6: The phases of the constant and linear isospin amplitudes.

	α_1	α_3	β_1	β_3	γ_3
phase	0.07	0.09	-0.12	-0.09	-0.03